

MATHEMATICAL RATIONALITY AND REALITY

Metaphysical and Theological Consequences

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ABSTRACT: Is formal mathematics a precise and objective instrument in order to express with objectivity our knowledge of the real world? The reduction of deductive reasoning to formal rules is a challenge which underlies the mechanistic ideal. Formal languages convert propositions into objects which can be handled by computers and have approximated human thought to that of computers. The results of formal mathematics cannot be complete and, therefore, they are open to several possibilities which cannot be predetermined in all cases. Another source of indeterminacy lies in the probabilistic and chaotic nature of the real world laws. This means that mathematical activity is, of necessity, faced with the risk of choosing from several possibilities. The opening up of mathematics to risk is not an opening up to irrationality. On asking meta-rationally for the permanence of global rationality, we verify that the consistency of the systems and the exclusion of mutual contradiction in their co-existence, is a meta-rational value, with metaphysical and theological consequences.

KEY WORDS: formal languages, consistent theories, completeness, real-world laws, meta-rationality.

Racionalidad matemática y realidad *Consecuencias metafísicas y teológicas*

RESUMEN: ¿Es la matemática formal un instrumento preciso y objetivo para expresar con objetividad nuestro conocimiento del mundo real? El uso de reglas formales en el razonamiento deductivo es un reto que subyace al ideal mecanicista. Los lenguajes formales convierten las proposiciones en objetos que pueden ser manipulados por computadores. Por otra parte, el desarrollo de la matemática ha mostrado que la matemática formal no puede ser completa. Otra fuente de indeterminación procede del carácter indeterminado y probabilístico de las mismas leyes del mundo real. Esto significa que la actividad matemática está necesariamente confrontada con el riesgo de elegir entre diversas posibilidades. Sin embargo, la apertura de la matemática al riesgo no es una apertura a la irracionalidad. Preguntándonos meta-razionalmente por la permanencia de la racionalidad global, observamos que la consistencia de los sistemas, entendida como exclusión de contradicción interna y exclusión de mutua contradicción, es un valor meta-razional cuya permanencia tiene consecuencias metafísicas y teológicas.

PALABRAS CLAVE: Lenguajes formales, teorías consistentes, completitud, leyes del mundo real, meta-racionalidad.

INTRODUCTION

What is reality? What role does mathematics play in the knowledge of reality? What vision does mathematics give us of reality? Frequently we consider that science brings us to the knowledge of objective reality as it is because science contributes what we call objective knowledge. Knowledge is objective when it is independent of the subject who knows. Objective knowledge is public and is not

taught in esoteric circles. It is taught at school and at university. Objective knowledge is independent of the oscillations of our subjectivity. Does science achieve its purpose of providing us with objective knowledge of reality? Is there scientific knowledge which is totally objective? As we shall see, the formal language of mathematics allows us to have a high degree of objectivity regarding the formulation of scientific knowledge. Is this objectivity total? To what extent can mathematics objectively formulate *all* scientific knowledge?

The first prerequisite for mathematics to formulate objective knowledge is that mathematics itself be objective. This will be the first question I pose in this paper, *Are the mathematical theories really objective and independent of the subject who formulates them?* As well as clarifying whether mathematics is objective, it is necessary to clarify the relationship between our mathematical knowledge and our knowledge of reality. The second question I will answer will be, *What is the relationship between mathematics and reality?*

The answers to these two questions will be paradoxical to some extent. I will show how the mathematical formulation is in fact a privileged resource which enables us to express knowledge of reality with a maximum degree of objectivity. However, at the same time, I will also show how knowledge formulated mathematically is plural, is open to a number of possible conceptions of mathematics, and, according to one conception of mathematics, it is open to the risk of choosing different hypotheses, a risk which is technically called undecidability, as it is based on the fact that we do not have reasons to decide on one hypothesis from several.

I will finish this paper by asking myself how the endeavours to ensure the clarification and rational objectivisation of mathematics as a formal science is projected on metaphysics. Metaphysics endeavours to reflect on the fundamental trans-experiential fundamentals of the knowledge of reality. The pluralism of formal systems and their inability to totally prevent the risk inherent to mathematically undecidable situations will lead us to ask about the rationality of inevitable decisions which mathematics forces us to take. If mathematics does not allow us to decide rationally in all cases, what rationality do we use in cases in which mathematics does not guide our decisions? The formal sciences lead us to a necessary opening up of knowledge to risk. This will be the third question in the paper, *In what sense can it be rational to assume the risk of committing errors?*

As a consequence of this approach, I will defend the legitimate relationship between reflection on formal sciences and theological reflection. For theology, God is the ultimate basis of reality. Metaphysics asks about the ultimate basis of reality. Theological reflection cannot avoid its relationship with metaphysical reflection as both ask about the ultimate basis of reality. If reflection on formal sciences leads us in the end to reflect on metaphysics, this reflection will affect theology. The fourth question in the paper will be, *What relationship is there between formal sciences, metaphysics and theology?*

1. ARE THE MATHEMATICAL THEORIES OBJECTIVE AND INDEPENDENT OF THE SUBJECT WHO FORMULATES THEM?

I will respond to this first question by reflecting on the historical evolution of mathematics, just as this evolution has occurred in the cultural context of modern science. We can say that the cultural context of modern science was created in Europe at the beginning of the XVII century around certain significant figures such as Galileo (1564-1642), Descartes (1596-1650), Newton (1643-1727), Leibniz (1646-1716)... Since then, modern science has continued to evolve up to the present time.

I will describe some epistemological features which have marked the cultural context of modern science, and will point out the singular role which mathematical formulation has had on scientific knowledge. I will then move on to focus on mathematics itself and will outline the historical process of the evolution of the epistemological conception of mathematics, pointing out several landmarks or historical moments in time which are keys to this evolution.

The cultural context of modern science: Empirical knowledge and mathematical formulation. Modern science arises from the formulation of empirical observations obtained by precise technical instruments and constructed in accordance with formal mathematical theories. In modern science both mathematical formulation and empirical observation are important. However, mathematical formulation is especially important as mathematics is not only used to formulate scientific theories but to design the observation devices. Galileo made a famous statement on the relationship between mathematics and empirical knowledge in 1610¹,

«Philosophy [nature] is written in this grand book which is always before our eyes, I mean the universe, but we cannot understand it unless we first learn the language and understand the symbols in which it is written. The book is written in mathematical language, and the symbols are triangles, circumferences and other geometric figures, and without their help it is impossible to understand even a word of it, and one wanders in vain through a dark labyrinth»².

On September 8, 1950, David Hilbert read a paper at the Congress of the Association of Scientists of Nature and Medical Doctors. In this paper, Hilbert, stated the importance of mathematical formulation for the specific, practical knowledge of reality, and as a protection for the autonomy of mathematics as a discipline independent of its applications,

«Mathematics is the instrument which links theory and practice, thought with observation. Thus, our culture, to the extent that it is based on intellectual knowledge and in the control of nature, is based on mathematics... in fact, we can say that we do not control a scientific theory of nature until we have extracted its

¹ KLINE, MORRIS, *El pensamiento matemático de la antigüedad a nuestros días*, vol. 1, Alianza Universidad, Madrid 2002, p. 434.

² KLINE, MORRIS, *Quoting Galileo*.

mathematical nucleus and have left it completely clean... however, despite this, mathematics has always rejected making its value depend on its applicability... the development of modern science depends on the development of the capacity for empirical observation and the capacity for mathematical formulation».

The historical evolution of mathematics. Science is not a static phenomenon. Science has been growing through several crises and changes. The philosopher Thomas Kuhn has described these changes as scientific revolutions³. Mathematics, as an essential part of science, is not a static phenomenon either. In recent centuries, mathematics has gained in objectivity. The mathematical propositions and theories are progressively expressed with more precision in a language which has become more and more clear and objective. I will now point out some periods of this *historical process* which led mathematics towards finding its more formal and abstract dimension, as objective, independent knowledge, or at least with the possibility of becoming independent of the empirical experience of the physical world. Throughout these periods, mathematics became a more objective and autonomous knowledge. In the XVII and XVIII centuries, mathematics seemed to be firmly established on the simple unshakable foundation of numbers and geometry. It was a clear language which referred to clear geometric and figures arithmetical signes. The foundations of mathematics were not questioned. Throughout the XIX century, mathematics was liberated from its dependence on the geometric-numerical intuition. One pioneering case of the liberation was the appearance of non-Euclidean geometries.

The overcoming of a naïve vision of geometric intuition. Since ancient times, geometric perception has been at the base of mathematics. Euclid (365-300? BC) in his *Elements* formulated fundamental propositions of Geometry in the form of axioms or postulates. Until the XIX century, the book of Elements of Euclid was paradigmatic as regards the study of Geometry. The fifth postulate of Euclid, as it appears stated in the *Elements*, states that «If a straight line crossing two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if extended indefinitely, meet on that side on which are the angles less than the two right angles». A shorter and equivalent way to state this postulate is owed to Proclo (411-485), «Through a point which is external to a straight line, it is possible to draw one and only one line parallel to this».

The intuition of the infinite. The fifth postulate differs from the other postulates in that it refers to the behaviour of straight lines to infinity. And the intuition of the infinite is conceptually separated from the intuition of finite objects. This has meant that its relationship with the other postulates has attracted geometers throughout history. Proclo (411-485) wrote a commentary to Euclid in which he attempted to derive the fifth postulate from the others.

Independence of the fifth postulate. The problem whether the fifth postulate of Euclid is an axiom which is independent of the others, that is to say, whether

³ KUHN, T., *The Structure of Scientific Revolutions*, Chicago, University of Chicago Press, 1970 [*La estructura de las revoluciones científicas*, México, FCE, 1971].

it is possible or not to deduce this from the other postulates, continued to intrigue geometricians. Among those who have studied this problem are the Italian Jesuit Saccheri (1667-1733) and the German philosopher Lambert (1728-1777). In his work *Euclides ab Omni Naevo Vindicatus*, Saccheri tried to prove the fifth postulate of Euclid supposing that it was false. He tried to derive a contradiction from the other postulates and its falsity. He would, thus, have proved that the fifth postulate is deduced from the other postulates, and, therefore, is not independent of these.

New perspectives in geometry. The Russian mathematician Nicolai Ivanovich Lobachevski (1793-1856), followed the route of Saccheri and Lambert, and assumed the hypothesis contrary to the fifth postulate of Euclid, that is to say, «through a point which is exterior to a straight line, it is possible to draw at least two lines parallel to this straight line». Based on this new postulate, in contradiction with the fifth postulate of Euclid, Lobachevski developed a new geometry without finding any contradiction. Lobachevski called his new geometry 'imaginary' as he did not find a «real» model for it.

The importance of logical consistency. One very important point to stress in the evolution of mathematical thought is that the strength of the argument of Lobachevski was not within the 'traditional' intuition of 'real' space, but within the logical consistency of his arguments. That is to say, in the absence of contradiction. Non-Euclidean geometry supposes that the fifth postulate of Euclid is false. The negation of the fifth postulate makes it possible to build 'several geometric worlds' and break the uniqueness of the intuitive world of classical geometry. Non-Euclidean geometry gives preference to logic over geometric intuition. However, from the fact that the arguments of Lobachevski might be consistent it was not deduced that non-Euclidean geometry was consistent, in the sense that a contradiction would not appear in it at some time.

Consistency of non-Euclidean geometries. Felix Klein (1849-1925) proved that non-Euclidean geometry is consistent (a contradiction cannot be deduced from its postulates) if Euclidean geometry is consistent. That is to say, he reduced the problem to proving the consistency of non-Euclidean geometry to the problem of proving the consistency of Euclidean geometry.

The connection between algebra and logic. The non-Euclidean geometries gave priority to the rigour of logical deduction as opposed to the geometrical intuitive perception of reality as had been usual until then. The rigour of logical deduction was not something new in the world of mathematics. The formalisms of mechanical deduction had been known since ancient times. The relationship between mathematics and the mechanist formalisms has its roots in history. At the start of the XIV century, the philosopher from Majorca, Ramón Llull, used complex mechanical techniques in his «Ars magna generalis summa», together with symbolic notation for the general formulation of knowledge. The start of modern science coincided with a more systematic use of algebra. The algebraic formalisms made it possible to mechanically structure the scientific observations which were shown in empirical experience, with a determinist character. The explosion of

mathematical research at the beginning of the XVII century was nourished from the point of view of algebra by two fundamental contributions⁴: 1. Systematisation as regards the handling of algebraic expressions. 2. The reduction of geometry to algebra by representing points of geometry by pairs of numbers. Descartes (1596-1650) and Fermat (1601-1665) are important at that moment. It was necessary to wait until the XIX century for George Boole (1815-1864) to apply algebraic methods to logic and develop logic as a part of algebra.

A formally mechanical deductive system. In the second half of the XIX century, Gottlob Frege (1848-1925) constructed the first formal logical system which includes all the deductive reasoning of ordinary mathematics. In 1879 Frege published *Begriffsschrift*, 'Concept Notation', with the subtitle, 'a language of formulas for pure thought in the likeness of arithmetic'. Frege intended that mathematics be constructed as a superstructure which had formal logic as its base. He introduced specific symbols for logical relationships in order to prevent confusion. He used quantifiers \forall , \exists . Concept Notation made it possible to represent the logical inferences as formal mechanical operations called rules of inference, which are based only on the way the symbols are placed.

At the end of the XIX century and the beginning of the XX century the development of mathematics as a formal objective system was sufficiently mature to be able to ask this question, Can mathematics justify itself as a purely formal science? The foundation of mathematics as a purely formal science would prove that mathematical knowledge is certain, secure, objective and public, and we can have complete confidence in mathematics. The reduction of deductive reasoning to formal rules is a challenge which underlies the mechanist idea of representing human reasoning through a formal mechanism. If this reduction were possible, we could entrust all reasoning to a mechanism which executes orders given in a formal language.

David Hilbert (1862-1943) and the meta-mathematical program. Meta-mathematics uses mathematics as a language to speak about mathematics as an object. In meta-mathematics the mathematical objects are represented by mathematical formulas, and these are reasoned from mathematics itself. The truth and falsity of mathematical theories is expressed by means of formal functions between these theories and its mathematical models. The meta-mathematical program intends to prove that the mathematical theories are *consistent*, *complete* and *semantically-decidable* by meta-mathematical finite resources. A theory is consistent when we cannot deduce a contradiction within it. A theory is complete if any true formula in the system can be formally proved from certain axioms. And a theory is semantically-decidable if we have a formal procedure for deciding whether a given formula is true in a given model or not. If a system is inconsistent, we can deduce any formula in it and the system will lose its logical value. If a system is incomplete, we will not be capable to derive formally from the given axioms all true propositions of the system.

⁴ DAVIS, MARTIN, *La computadora Universal*, Ed. Debate, Madrid 2002, p. 22.

Mathematical foundationalism of Hilbert. Hilbert intended to establish mathematics as a basis of certain and objective knowledge. If the meta-mathematical program was a success, the consistent, complete and decidable systems of mathematics would provide a secure instrument for science to access reality. The security of mathematical knowledge would be based on epistemological certainty transmitted by certain *signs* whose properties are described by the basic axioms of mathematics. In a course given by Hilbert on 'Elements of Euclidean Geometry' in 1898, Hilbert stressed that it should be shown that geometrical theorems are deduced from certain axioms through pure logic with no dependence on geometric intuition, and with this he highlighted the logical-formal value of mathematics. According to a famous anecdote, the theorems had to continue being valid if, instead of points, lines and planes, we speak of 'tables, chairs and glasses of beer', on condition that it is supposed that these objects obey these axioms.

Crisis of the Hilbert programme. The theorem of completeness and the theorems of incompleteness of Gödel. In his doctoral thesis Kurt Gödel (1906-1978) proved that all the propositions logically valid, that is to say, all the logical propositions which are true in all the models can be obtained from a system of logical axioms. This result is called 'completeness of the logic of predicates of first order'. Once Gödel had proved that logic was a complete system. The next step was to prove that arithmetic was a complete system. However, later on, by using meta-mathematical methods, Gödel proved that the formal system of arithmetic is incomplete. That is to say, regardless of the fact that additional first order axioms are added to the system of arithmetic on condition that the new axioms do not lead to a contradiction (i.e. a proposition of the type $A \wedge \neg A$), there will be a proposition U which will be undecidable in the system of arithmetic. That is to say, within the formal system of arithmetic, U cannot be deduced from the axioms. Therefore, it is not possible 'to decide' whether U belongs to the system or not. Gödel also proved that, if arithmetic is consistent, then it is not possible to prove within the system of arithmetic the formal proposition which expresses the consistency of arithmetic.

The incompleteness of arithmetic proves that we cannot construct an axiomatic system from which we can deduce all the propositions valid in arithmetic. From any axiomatic system we can only deduce part of the arithmetic. This can be explained graphically, one day we go into a shop and, in order to know the price of a purchase we have made, we make some arithmetical calculations. Another day, we are faced with more complicated arithmetical calculations and we also resolve these. It is as if we were in the dark room of arithmetic when we make arithmetical calculations and we illuminate part of the room with a lantern and resolve our problem regarding a particular object in the room in each case. The problem of the completeness of arithmetic consists of asking if there is a way to illuminate *all* the «arithmetic room» at the same time. Incompleteness tells us that it is not possible to illuminate everything through an axiomatic system. Any illumination we make will be partial, just as the illumination of a lantern is partial.

It is important to point out that the proof of incompleteness of arithmetic presupposes that arithmetic is consistent. The proposition obtained by Gödel is conditional: if arithmetic is consistent, then it cannot be complete. Consistency remains as a value of any logical-mathematical proposition which cannot be renounced. Moreover, from the incompleteness of arithmetic it is deduced that we cannot prove the consistency of arithmetic from within the arithmetic. That is to say, in order to prove the consistency of arithmetic, we need to be supported by another meta-theory external to arithmetic, whose consistency we must also presuppose.

Pluralism in the conception of mathematics. According to the classical conception, a mathematical proposition is either valid or it is not. This classical principle is frequently called in Latin the principle of 'tertio excluso'. According to this principle, in order to prove the validity of a proposition, it is sufficient to prove that a contradiction is deduced from its negation. Furthermore, in the mathematics we call classical, the existence of infinite sets is admitted. This classical way of reasoning is at the base of the reasoning of Hilbert and Gödel. When the programme of Hilbert entered a crisis, the classical conception of mathematics also entered a crisis. There are other conceptions of mathematics such as the intuitionist/constructivist conceptions, which have a stricter attitude and only admit propositions which are effectively proved as valid. Depending on the schools, this conception leads them to different types of restrictions as regards the existence of sets with infinite elements as we cannot construct an infinite set effectively. Outstanding among the constructivist mathematicians is the Dutch mathematician L. E. J. Brouwer (1881-1966). The incompleteness of arithmetic had proved the incapacity of classical arithmetic to completely describe all the mathematical objects, Brouwer renounced presenting the problem of completeness in the classical fashion and developed a stricter vision of mathematical reasoning. For Brouwer only the propositions obtained through an effective proof referring to finite objects we have a direct intuition of are valid. Both conceptions of mathematics can coexist separately. The propositions which have a constructive proof are also classically valid. However, a constructivist will not admit the logical axiom of the 'tertio excluso'. Thus, it would be contradictory if we introduced both conceptions within the same system. The existence of separate systems based on different conceptions of logic is not contradictory. Using a meta-language based on classical logic, we can reflect on constructive logic. From the outside, we can see that both logics have different approaches but neither of them proves the falsity of a theorem of the other. Moreover, we can consider constructive mathematics as a subset (the constructive part) of classical mathematics.

Pluralism in the conception of mathematical objects. Sets, Structures and Categories. Pluralism has been extended to the same conception concerning what the mathematical objects are. Hilbert and Gödel described the mathematical objects based on the theory of sets of Cantor. The theory of sets makes it possible to give a semantic definition of the validity of a mathematical proposition. Using sets, we can say that a mathematical proposition is true in a certain set if the com-

ponents of this set fulfil the relationships which appear in the proposition, and if they are not fulfilled, we say that the proposition is false. We can specify this with an example, we say that the proposition ' $2 + 3 = 5$ ' is true in the set of natural numbers because the addition function is interpreted in the set of natural numbers and is applied to the sets which we designate as '2' and '3' and returns the set which we designate as the number '5'. It is interesting to note that subsequently other theories have also been developed and these started from other abstract and general descriptions of mathematical objects other than sets. Thus, Bourbaki attempted to construct all modern mathematics based on the concept of structure. Later, the concept of category appeared as a basic description of mathematical objects, categories are not based on the concepts of sets and belonging, but on the concepts of function and composition.

The plural coexistence of contrary formalisms. Classical mathematics and constructivism are formalisms with incompatible approaches. It would be contradictory to admit that a system admits the principle of 'tertio excluso' and rejects it at the same time. But we can speak of both types of mathematics and compare them from the outside through a meta-language. The comparative meta-study of incompatible systems makes it possible to isolate each system and interrelate the systems from the outside. This is like a cluster of grapes where each seed is a system. Each system has internal consistency. We cannot unite two systems, making one system from two. This would be contradictory and we would destroy them. However, we can unite them and form a cluster. The branches are the meta-language. Each system is consistent and the cluster is also consistent, but the systems must remain separate in order to maintain their consistency. The meta-language allows us to have a vision of the plurality of formal systems which is simultaneously open and consistent.

The problem of decidability. Formal logic and the mechanist ideal. The incompleteness of arithmetic was a failure of the mechanist ideal, but the approach to the problem of decidability and its resolution would, in some way, involve a deepening of this failure. The mechanist ideal consisted of obtaining a mechanical artifice by which it would be possible to execute all the deductions. The development of formal logic was a great step forward on the way to achieving the mechanist ideal. The deductions carried out through formal propositions can be executed by a machine. Through the theorem of completeness of the first order logic of predicates, Gödel had proved that any correct deduction could be formally executed through the rules of deduction of the logic of predicates. Once this problem was resolved, another important problem remained to be resolved, the problem of decidability (*Entscheidungsproblem*): given some premises and a conclusion, is it possible to effectively decide whether the conclusion is deduced from the premises?⁵, that is to say, can we mechanically decide whether a proposition belongs to a certain theory, by using the rules of deduction of the logic of predicates of the first order? The mechanist ideal was previous to and more

⁵ DAVIS, MARTIN, *La computadora Universal*, Ed. Debate, Madrid 2002.

general than the specific procedures of formal logic and the reasoning of the theorem of incompleteness of Gödel, but the mechanistic artifice which sought the mechanistic ideal had never been formally specified in history. Gödel had used mechanical procedures, but when Gödel wrote his theorem of incompleteness, there was still no general definition of a mechanical procedure in general.

The informal idea of a mechanical artifice. Informally the idea of a mechanical artifice for deduction was present in the logical and algebraic procedures. An 'effective deduction' is characterised by four properties. 1. Its conduct is governed by a finite number of precise instructions (in computers this set of instructions is called a program). 2. The artifice is capable of executing these instructions in a finite number of steps (in computers we need that the processes terminate). 3. The execution of these instructions does not permit any type of initiative by the artifice which executes the instructions. (The instructions are executed mechanically). 4. A human being who had sufficient time could simulate the execution of these instructions using a pencil and paper (the execution can be represented formally). These are the informal characteristics of the mechanical procedure with which some logicians and philosophers had dreamed would have been able to make all the deductions.

The informal idea of a mechanical artifice corresponds with the informal idea of the algorithm traditionally present in algebra. From ancient times, these four characteristics of the 'effective method' were used informally and not rigorously in order to characterise the algorithmic processes of algebra and formal logic, which was already rudimentarily known by Aristotle. The key requirements of the 'effective method' are the finiteness of the processes and the mechanical or ingenuous execution of the processes, lacking any intention or initiative alien to the method.

Alan Turing (1912-1954) specified the informal idea of the mechanical artifice through what we call the Turing machine. In his paper in 1936⁶ Turing presented an exact formalisation of the informal concept of the 'effective method'. Alonzo Church (1903-1995) had presented another different formalisation some months previously⁷. The formalisations of Turing and Church were different but turned out to be equivalent in the sense that both described the same set of procedures or functions. The Turing-Church thesis states that both sets of functions, the one defined by Turing and the one defined by Church, contain exactly the set of functions whose value can be calculated by what had until then been called the 'effective method'.

The Church-Turing thesis and mechanicism. The Church-Turing thesis⁸ states that the Turing machine and the lambda calculus can represent any formal effective

⁶ TURING, A. M., «On Computable Numbers, with an Application to the Entscheidungsproblem», *Proceedings of the London Mathematical Society*, series 2, 42 (1936-37), 230-265.

⁷ CHURCH, A., «An Unsolvable Problem of Elementary Number Theory», *American Journal of Mathematics*, 58, 345-363.

⁸ Alonzo Church (1903-1995), developed the lambda calculus, and proved the existence of undecidable problems before his student Alan Turing proved the existence of unsolvable problems through his machine.

tive computation process. Although the Church-Turing thesis is not proved, its evidence is commonly accepted by mathematicians. It is commonly accepted that any formal algorithmic process which we can carry out can be represented either of the mechanisms.

The undecidability of the formal systems. The halting problem or the problem of detainment for Turing machines is the most well known example of the problem of undecidability. It consists of deciding if a Turing machine will stop or it will remain in an infinite cycle. Once the mechanism of his machine was established, Turing obtained a surprising result which strengthened the theorem of incompleteness of Gödel. Turing proved that it is not possible to formally proof in all cases whether a given program will stop or not.

Applied formalisms. The ideas of Church and Turing are at the core of current theoretical computing. Moreover, the applicability of computing has decisively affected the development of the applied dimension of the formal sciences. It has highlighted applied dimensions, unknown until now, of languages and formal models such as: pragmatism, experimentation, implementability and efficiency. This has led to the plurality of logics, suited to several finalities. The existence of a plurality of logics opens up new perspectives for scientific language as each of these reflects a partial dimension of reasoning. In addition, on applying its formalisms, computing has discovered new probabilistic properties of algorithms.

The problem of complexity. In search of the best solution. In plural and undecidable mathematics we are forced more and more to find different solutions to the same problem. We need a criteria to determine which is the best solution among these. The best solution will be the simplest. The least complex.

Computing and complexity. Half way through the sixties, A. N. Kolmogoroff and Gregory J. Chaitin, independently, developed what Chaitin called «algorithmic theory of information» which measures computational complexity. Given that computing is formal mathematics applied to the development of algorithms, two factors are very important: the time a machine requires to carry out a calculation, and the size of a program, that is to say, the amount of information which has to be communicated to a computer so that it can carry out an operation. Chaitin relates the algorithmic theory of information with the entropy which measures the level of disorder of a system. For Chaitin the size of a computer program is analogous with the degree of disorder of a physical system:

«It is sufficient to think of the principle of "Occam's razor": the simplest theory is the best one. What is a theory? It is a computer program for the prediction of observations. The affirmation that the best theory is the simplest one becomes the affirmation that a concise computer program constitutes the best theory. And if there is no concise theory? And if the shortest program capable of reproducing a set of experimental data is the same size as the set of data? In this case, the theory is of no use, it is a performance, the data is incomprehensible, random. A theory is only good insofar as it compresses the data to the point of creating a system of theoretical hypotheses and rules of deduction, which is much smaller. Thus, we could define random as what cannot be compressed. The only way to

describe an object or number which is completely random to a person is to show it to him and say, "Here it is"»⁹.

Complexity and incompleteness. The theorem of incompleteness of Gödel appears again. The study of the complexity of a program is necessarily incomplete:

«The complexity of something is measured by the size of the minimum program of the computer which makes it possible to calculate it. However, how can we be sure that we have the minimum program? The answer is that we cannot. This is one of my favourite propositions on incompleteness, If we have n bits of axioms, it will never be possible to prove that a program is the smallest possible if its size exceeds n bits... The set of axioms which mathematicians normally use is quite concise, if this were not so, nobody would believe in them. In practice, there is a vast world of mathematical truths, an infinite amount of information, while any set of axioms only takes in a finite, minute amount of this information. This is why, in short, the theorem of incompleteness of Gödel is not mysterious and complicated, but natural and inevitable»¹⁰.

While the incompleteness of arithmetic is an abstract result of pure mathematics, the study of the complexity of the programs is practical and applicable. In computing, a direct application of the mathematical formalisms to the processing of the data of the empirical sciences occurs.

Answering the first question about the objectivity of mathematical theories and their independence of the subject who formulates them, we can say that mathematics are both, objective and independent of the subject because they are formal, and non objective but dependent of the subject who formulates them, because they are incomplete and undecidable.

2. WHAT IS THE RELATIONSHIP BETWEEN THE MATHEMATICAL THEORIES AND REALITY?

When answering the question on the objectivity of mathematical knowledge, we have seen that the incompleteness and the undecidability of the mathematical systems encouraged the pluralism of these systems. The undecidability and the plurality means that the mathematical systems cannot be made independent of the human mind which creates them. At this point, we can look towards the world of empirical observations and ask ourselves: How does mathematics help the human mind to draw a real image of the universe, just as the universe is? I will refer to three questions which relate mathematics and reality: *a)* The intelligibility of reality. *b)* The intelligibility of reality within a chaotic and probabilistic order. *c)* The coexistence of reason and risk. The inevitable conflict.

⁹ CHAITIN, GREGORY J., «Ordenadores, paradojas y fundamentos de las matemáticas», *Investigación y Ciencia*, July, 2003.

¹⁰ Ibidem.

a) *Intelligibility of reality. Empirical observation and mathematical theories.* We can say that, throughout the XX century, the ingenuous scientific realism which established a simple, naïve relationship between the knower and the known object entered a crisis. In the case of quantum physics, we know that the empirical observations are not totally objective, they are affected by the observer. However, these quantum observations are expressed in mathematical theories which pretend, in some way, to be objective, that is to say, independent of the subject who formulates them. Quantum physics is particularly intelligible when we discover laws in it which we can express with mathematical rigour. This is how modern science has understood it since its origins. Moreover, this mathematical formulation was sought and achieved over recent centuries in several fields of scientific knowledge. The intelligibility of reality is manifested in the mathematical consistency of formal structures through which we interpret the laws of empirical reality. Mathematics is applicable because it is capable of formally formulating the empirical laws which govern in nature, regardless of whether these are determinist or indeterminist, certain or probable.

Two visions of mathematics. There are two opposing views of mathematical knowledge considered in itself. One of these considers mathematics to be a discovery, that is to say, as a result of the description of worlds and realities which exist in themselves and which the mathematician finds outside himself. This view is often referred to as Platonism. The other view considers that mathematics is pure creation. According to this second conception, mathematics is a formal world created by mathematicians.

Two attitudes of mathematicians faced with reality. Among the mathematicians there is an applied attitude which pretends to use mathematics as an instrument to formally describe and explain the laws of the empirical world which we find in the natural sciences. Another attitude considers mathematics to be a pure science, an art through which the mathematician creates theorems and other mathematical objects with no concern for its practical application; just as the musician creates symphonies and other musical pieces, and the poet writes poetry.

Applied mathematics is in the origins. Within the historical origins of mathematical activity, in the Egyptian and Mesopotamian cultures, we find mathematical applications to specific problems such as the calculation of weights and measures in commercial transactions, and the resolution of geometric problems concerning the measurement of fields and surfaces. Modern physics encouraged the applied activity of mathematics through the use of functional analysis, differential calculus, tensor calculus, etc. In the physics of the XX century mathematics was an essential instrument for the formulation of the theory of relativity, quantum mechanics and finally string theory. Above all, throughout the XX century, new empirical knowledge which traditionally was not formulated with mathematical rigour and, therefore, was not considered to be scientific knowledge in the rigorous sense of modern science has been progressively formulated mathematically. These new mathematical formulations have given rise to the appearance of new rigorous sciences such as economics, biology, geolo-

gy, etc., which use mathematical models and methods such as game theory or optimisation. In recent decades, the capacity for the formalisation of empirical observations has been developed substantially thanks to computing. Computing has also served to find practical applications for the totally abstract and formal formulations of programming languages.

Alfred North Whitehead describes what the mathematical process of abstraction must have been like in its origins. The perception of numbers and geometric figures, as these are shown in reality, are the origin of formal mathematics.

«Suppose we project our imagination backwards through many thousands of years... During a long period, groups of fishes will have been compared to each other in respect to their multiplicity, and groups of days to each other. But the first man who noticed the analogy between a group of seven fishes and a group of seven days made a notable advance in the history of thought. He was the first man who entertained a concept belonging to the science of pure mathematics»¹¹.

Mathematics has traditionally studied numbers and geometric figures. At the present time, it studies all kinds of relationships more abstractly, including the logical relationships between sets of premises and the conclusions which can be inferred from these:

«Mathematics is thought moving in the sphere of complete abstraction from any particular instance of what it is talking about. This view of mathematics is so far from being obvious that we can easily assure ourselves that it is not, even now, generally understood. For example, it is habitually thought that the certainty of mathematics is a reason for the certainty of our geometrical knowledge of the space of the physical real world. This is a delusion which has vitiated much philosophy in the past, and some philosophy in the present»¹².

Applied mathematics describes the world which is shown to the scientist as intelligible, thus, we can say that applied mathematics refers to the ontology of reality. Empirical and technological science are possible because there is a correspondence between the laws of the real world and formal mathematics. The mathematical formulation of the laws of the real world make it possible to design machines and instruments which permit the technological transformation of reality. As it is abstract, formal mathematics is, in a way, independent of scientific observation, but scientific observation and technological applications require mathematical language in order to express their intelligibility and their capacity to control reality.

Consistency and empirical reality. The consistency of the formal systems enables us to make predictions on the conduct of empirical reality. The consistency of the formal systems applied to the explanation of the empirical reveals the internal consistency of the empirical data and enables technological control and prediction. A law in a consistent system serves to predict whether a certain

¹¹ WHITEHEAD, ALFRED NORTH, 1861-1947, *Science and the Modern World* (Chapter 2), Cambridge [England]: The University Press, 1953.

¹² Ibidem.

event A will or will not occur. If the system does not behave consistently either A or not A could occur. If a formal system is consistent, it is apt for technological application, even though this application is not known. For example, non-euclidean geometry was a consistent system with no important applications before its use in the explanation of the theory of relativity.

b) *The intelligibility of reality within a chaotic and probabilistic order.* However, reality is reluctant to allow itself to be controlled. Reality resists when mathematics attempts to enclose it within determinist orders. Until now, we have seen that the intelligibility of mathematics is necessarily undetermined as a consequence of the internal nature of formal systems. Now we take a step forward. In the chaotic and probabilistic orders, we find that, in the empirical observation of reality indeterminacy appears together with determination, unpredictability appears together with predictability.

In the probabilistic order, empirical events occur independently of each other, and therefore each of these is unpredictable in relation to the others. Nevertheless, together they comply with the probabilistic laws which are represented with a mathematical order.

We empirically observe systems with chaotic conduct which, however, have an evolution which leads to the same 'attractor' from different starting points. The structure of an attractor can be determined or be probabilistic. In the probabilistic conduct of attractors, we find an order within disorder which in a certain way increases the enigma of intelligibility in the world. We also observe events with chaotic determinist conduct which are very sensitive to the initial conditions so that a small variation in these produces a substantial change in the trajectories. The trajectory of a chaotic system can converge toward an 'attractor' which may be a relatively simple structure or it may converge towards a strange attractor with a probabilistic structure. The attractors also represent order within disorder.

However, the probability does not only appear in the empirical world, the internal laws of arithmetic itself have been shown to be random. For example, there are statistical regularities which make it possible to forecast the average distribution of prime numbers, while the position of each prime number in particular seems to be random. The probabilistic order appears both in the pure formal sciences and in empirical observations.

The appearance of chaotic and probabilistic orders, which permit degrees of liberty for particular events within the general laws, transforms the view of rationality and the view of the world. The rationality of the empirical observations goes from being mechanical to being undetermined and probabilistic.

c) *Coexistence of reason and risk. The inevitable conflict.* Remembering Galileo, we can say that nature is written in a rational language. However, unlike Galileo, we do not reduce this language to triangles, circumferences and other geometric figures. We can say that the conception which science has of the world has evolved because the scientific axioms have lost their character of unique, definitive explanations. This has meant that the value of scientific propositions has become more relative and their importance is more conditioned by the value of their technological application.

At the present time, science is characterised by a grand capacity to create diverse models which describe different and autonomous aspects of the world, and diverse scientific communities base their theories on these models. The scientific communities are autonomous of each other. Cosmology bases its theories on the origin and development of the world in diverse models of the Universe. Microphysics develops models which describe the ultimate constitution of matter. Biology applies its models to describe other dimensions of the world. In scientific psychology, we find models which describe the conduct and the nature of men and women. The very history of the different scientific models is scientifically modelled.

Throughout the XX century, substantial efforts were made to create formal languages. The formal language converts propositions into objects which can be handled by computers and has approximated human thought to that of computers. On February 20, 1947, Turing gave a lecture in the London Mathematical Society where he asked to what point it was, in principle, possible that a computer might simulate human activity. This led him to proposing the possibility of a computer which was programmed to learn and which would be allowed to commit errors. «There are several theorems which state almost literally that [...] if a machine is expected to be infallible, then it cannot be intelligent [...]. However, these theorems do not say anything about how much intelligence a machine which does not pretend to be infallible must exhibit»¹³. Given that we cannot deduce all the propositions of a system of axioms which can be manipulated, it is necessary to try out several propositions and discard them if they do not serve. Turing concluded his lecture with a call for 'fair play with computers', and these should not be expected to be more infallible than human beings. He also suggested that chess could be a suitable exercise to begin with¹⁴.

The fair play which Turing asked for consists of accepting the risk of making certain propositions which, even in the simple case of the game of chess, are not deduced from other accepted ones. There is no sufficiently complex formal game which does not require risk.

Leibniz proposed that, in order to terminate conflicts, when two persons were involved in litigation, they should define the concepts within a formal system of calculation. The parties involved in discussion had simply to sit down and simply make calculations. Leibniz thought that all conflicts could be resolved in this way. Nowadays, we see Leibniz's proposal as naive.

The risk of committing errors. At the same time that the meta-mathematical attempt to base formal mathematics from formal mathematics itself has been intellectually very fruitful, it has also shown that formal-mathematical thought is necessarily open to the risk of committing errors. I do not refer to the errors which originate in sensorial perception, or in the measuring devices. I refer to the opening up to the risk which is intrinsic to the formal systems themselves and which is deduced from the undecidability of these systems.

¹³ DAVIS, MARTIN, *La computadora Universal*, Ed. Debate, Madrid 2002, p. 222.

¹⁴ TURING, A., *Collected Works. Mechanical Intelligence*, Amsterdam, North-Holland, 1992.

3. IS IT RATIONAL TO ASSUME THE RISK OF COMMITTING ERRORS?

If all the situations involving a possible choice are predetermined, free will is impossible. For free will to exist, there must be indeterminacy and risk in some way. Science discovers more and more environments in which new determinations which were unknown before appear. The progress of the study of genes and the influence of the environment on conduct mean that we currently consider actions as predetermined and before we had considered these to be free. Therefore, it is legitimate to ask ourselves whether all reality will be determined and there is no possibility of free will.

Indeterminacy in the formal sciences. The innovation brought about by the theorems of undecidability and incompleteness is that we are faced with risk in the very environment of formal sciences, where we can exercise most control over our knowledge. Formal logic permits a total control of knowledge, insofar as it is based on the execution of instructions which do not permit any kind of unprogrammed initiative by the person who executes these. However, we know that any computer, or any human mind which wants to have the capacity to formally express all its thoughts, would have to assume the risk of committing errors. Precisely because this risk of committing errors is based on the very nature of the formal processes, there is no formal explanation which tells us how we have to assume this risk.

Meta-rationality and consistency. Given that the formal sciences cannot avoid this risk, we are now led to ask about the rational presuppositions which justify the situations of indeterminacy and risk where we are taken by our current concept of formal science. Mathematics has renounced being complete and has also renounced that all its important systems are decidable. Consistency is a meta-rational pretension. Although the systems are plural, each one of these must have internal consistency and meta-reflection about the different systems need to be consistent. The formal systems may be different and independent, and use different logic, but there is a pretension that, in a way, is absolute and common to all the formal systems: its internal consistency and the consistency of their co-existence. The formal systems are justified by their internal consistency, and their plurality is justified by the consistency of their co-existence. There is no possible alternative to consistency. If we state A and not A at the same time and exactly in the same sense and referring exactly to the same reality, we have lost rationality. Consistency is a value of rationality which cannot be renounced. We can prove the consistency of a system in particular and there may be systems whose consistency we cannot prove, but the consistency of the systems is a presupposition which we cannot renounce.

Mathematics and consistency. It would be very important to be able to prove that mathematics is consistent. In 1936, Gerhard Gentzen (1909-1945), using transfinite induction, therefore, from outside arithmetic proved that arithmetic is consistent. However, as stated by E. B. Davies «*If one proves the consistency of arithmetic by invoking some other, richer, formal system, one achieves nothing unless*

one considers that the consistency of that new system is less capable of being doubted»¹⁵. It is normal that the consistency of a more complex system than arithmetic is more questionable than the consistency of arithmetic. What would occur if arithmetic were inconsistent? Arithmetic is the most basic formal system of mathematics. If arithmetic were inconsistent, the arithmetical reasoning would only be valid so long as we do not find any inconsistencies in it. Thus, the value of arithmetical reasoning would cease to have *absolute* value and have only *local* value, it would have value only in its consistent subsystems. The mathematical derivations would continue to be valid in order to establish equivalencies between formulations, equations and theories, and would also serve to predict properties of computer programs. Mathematics would maintain its current value on the condition that it is limited to mathematical arguments and computer processes within the limits in which the systems, or parts of the systems, maintain their consistency.

4. WHAT RELATIONSHIP IS THERE THEN BETWEEN THE FORMAL SCIENCES, METAPHYSICS AND THEOLOGY?

Traditionally metaphysics is the discipline which reflects on the ultimate questions about reality. Metaphysics intends to reflect on ultimate questions for global reality with objectivity, that is to say, regardless of the subject who is reflecting. The meta-rational justification of the formal sciences we seek is metaphysical because it intends to be objective and because we use it to ask ourselves about the ultimate rational justification of our formal thought.

The human mind needs to find sufficient reasons which consistently explain its experience of the real world. When the empirical sciences use mathematics as a means of expression, they use its capacity to express data objectively and establish causal relationships between this data in order to consistently explain their experience. The consistency of the explanation is a presupposition which cannot be renounced, in the sense that a contradiction cannot appear in the explanation.

However, mathematics applied to the formalisation of reality cannot provide a sufficient reason for all experiences. There are two facts which show the intrinsic limitation to the mathematical explanation: *a)* At the same time as mathematics gives a mechanical and determinist explanation, it shows the insufficiency of this explanation. *b)* Nor can mathematics give a reason why it is a language which contacts reality. Is mathematics a mask which disfigures the real world? How does mathematics access the real world? We do not have a formal explanation which explains how mathematics accesses the real world. In the same way as we presuppose meta-rationally that the systems are consistent while not otherwise proved, we also need to presuppose that the formal systems describe laws which occur in the real world.

¹⁵ DAVIES, E. B., «Gödel, Inconsistency, Probability, and Truth: An Exchange of Letters», in: *Notices of the AMS*, April 2006, p. 463.

Theology reflects on God as the ultimate foundation of reality. Theological reflection cannot avoid its relationship with metaphysical reflection as both ask about the ultimate questions of reality. Different ways of understanding theological reflection correspond to different metaphysical conceptions of reality. Leibniz had a deterministic conception of rationality and, consequently, he had a deterministic vision of theology. At the present time, we do not have a deterministic vision of rationality and this also influences theological reflection. Moreover, if it is considered that the principle of consistency of the systems is a meta-rational principle which cannot be renounced, theological theories need necessarily to co-exist consistently with scientific theories.

CONCLUSION

Mathematics is a historical science which evolves and is developed in the same way as the other sciences. A feature of mathematics is that it has the maximum capacity known to express its results objectively through a formal language and precise laws of deduction. In fact, historically mathematics has progressively developed its capacity to objectively express its results. Thus, mathematics is the most precise and objective instrument we have in order to express our knowledge of the real world.

The development of mathematics throughout the XX century proved that the results of mathematics cannot be complete and, therefore, they are open to several possibilities which cannot be predetermined in all cases. Another source of indeterminacy lies in the laws of the real world which have a probabilistic and chaotic nature. This means that mathematical activity is, of necessity, faced with the risk of choosing from several possibilities.

The opening up of mathematics to risk is not an opening up to irrationality. The fact that we are open to risk does not prevent us from asking about the rationality of this situation. On asking meta-rationally for the permanence of global rationality, we verify that the consistency of the systems, as the exclusion of internal contradiction and the exclusion of mutual contradiction in their co-existence, is a rational value which must continue. The consistency of the systems may or may not be proved formally, but this is a presupposition which we cannot renounce.

The consistency of the systems acquires metaphysical value as the ultimate basis of rational thought, and, of necessity, it also has theological value. All theological theories need to be consistent with the scientific theories. The dialogue between science and religion will, above all, be based on consistency, as the exclusion of contradiction, between the propositions of science about the reality and those of theology.

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